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β 级 α 型 λ-Bazilevič 函数的对数系数

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摘要: 利用从属关系给出 $|(g(z)/f(z))^{\alpha}|$ 的估计,并运用构造一个非负函数和对复变函数模的积分进行估计的方法,研究 β 级 α 型 λ-Bazilevič 函数类 $B(\lambda,\alpha,\beta)$ 的对数系数 b_n,得到 $|b_n|\leqslant(A\log n)/n+B/n+32\beta/(1-|1-2\beta|)$,其中 A、B 是绝对常数,推广了相关结果.

关键词: 单叶函数; 对数系数; Bazilevič 函数
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The Logarithmic Coefficients of λ-Bazilevič Functions of Type α and Order β

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Abstract: Estimation of $|(g(z)/f(z))^{\alpha}|$ is given by using subordination. Using the method of construction a non-negative function and estimation the integration of model of a complex function, the logarithm coefficient b_n of λ-Bazilevič functions of type α and order β is studied. The estimate $|b_n|\leqslant(A\log n)/n+B/n+32\beta/(1-|1-2\beta|)$ is given, where A,B are absolute constants. The results are generalized.

Key words: univalent functions; logarithmic coefficients; Bazilevič functions

设 S 表示在单位圆盘 $U=\{z:|z|<1\}$ 内单叶解析函数 $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ 构成的函数类,用 S^*、C 和 B_α 分别表示通常的星象函数类、近于凸函数类和 Bazilevič 函数类,用 Y 表示圆对称函数类[1],

β 级 α 型 λ-Bazilevič 函数类 $B(\lambda,\alpha,\beta)$ 定义如下:

定义 1[2]　设 $f(z)\in S,\alpha\geqslant0,0\leqslant\beta<1,\lambda>0$,若存在 $g(z)\in S^*$,使得

$$\mathrm{Re}\left\{(1-\lambda)\left(\frac{f(z)}{g(z)}\right)^{\alpha}+\lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}\right\}>\beta,\quad(1)$$

则称 $f(z)\in B(\lambda,\alpha,\beta)$.

显然 $B(1,\alpha,0)$ 为熟知的 Bazilevič 函数类 B_α. Bazilevič 函数是复变函数论中一类非常重要的单叶函数,学者们从不同的角度出发,研究了一些有趣的 Bazilevič 函数子类[3-9].

设 $f(z)\in S$,若

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}b_nz^n,\quad(2)$$

则称 b_n 为 $f(z)$ 的对数系数. 对数系数的估计在单叶函数的系数估计中有重要作用. Keobe 函数 $k(z)=z(1-z)^{-2}$ 的对数系数为 $b_n=1/n$. 对 b_n $(n\geqslant2)$ 的估计结果[10-14]如下:当 $f(z)\in S^*$ 时,$|b_n|\leqslant1/n$;当 $f(z)\in C$ 时,$|b_n|\leqslant(A\log n)/n$;当 $f(z)\in B_\alpha$ 时,$|b_n|\leqslant(A(1+\alpha)\times\log n)/n$;当 $f(z)\in Y$ 时,$|b_n|\leqslant(A\log n)/n$;当 $f(z)\in B(\alpha,\beta)$ 时,$|b_n|\leqslant(A\log n)/n$. 以上的 A 表示不同的绝对常数.

本文将运用构造一个非负函数和对复变函数模的积分进行估计的方法研究 β 级 α 型 λ-Bazilevič 函数类 $B(\lambda,\alpha,\beta)$ 的对数系数 b_n.

1　引理

引理 1[14]　设 $f(z)\in S$,则

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}=\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right)+1.$$

引理 2[14]　设 $f(z)\in S,\alpha\in\mathbb{C}$,则对 $z=re^{i\theta}$ $(0<r<1)$,有

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$$\frac{\partial}{\partial\theta}\left(\arg\left(\frac{f(z)}{z}\right)^{\alpha}\right)=\alpha\frac{\partial}{\partial\theta}\left(\arg\frac{f(z)}{z}\right).$$

引理 3[11]　设 $f(z)\in S$,则对 $z=re^{i\theta}$ $(1/2\leqslant r<1)$,有

(1) $\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2\mathrm{d}\theta\leqslant 1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}$;

(2) $\frac{1}{2\pi}\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2\mathrm{d}\theta\mathrm{d}r\leqslant 1+2\log\frac{1}{1-r}$.

引理 4[15]　设 $g(z)\in S^*$,则

$$\frac{\partial}{\partial\theta}\arg g(z)>0,\int_0^{2\pi}\frac{\partial}{\partial\theta}\arg g(z)=2\pi.$$

引理 5[16]　设 $f(z)\in S$,则对 $0<r<1$,有

$$\sum_{k=1}^{\infty}k|b_k|^2r^{2k}\leqslant\log\frac{1}{1-r}.$$

引理 6　设 $f(z)\in B(\lambda,\alpha,\beta)$,$g(z)\in S^*$ 满足式(1),则对 $z=re^{i\theta}$ $(0\leqslant r<1)$,有

(1) $J_1=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Re}\frac{zf'(z)}{f(z)}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}\theta\right|\leqslant 3$;

(2) $J_2=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Im}\frac{zf'(z)}{f(z)}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}\theta\right|\leqslant 3+4\alpha\left(1+2\log\frac{1}{1-r}\right)$;

(3) $J_3=\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}e^{in\theta}\mathrm{d}\theta\right|\leqslant 4\alpha\times\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}$.

证明　(1)由引理 1 和引理 2 可知

$$J_1\leqslant\frac{1}{2\pi}\left|\int_0^{2\pi}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{1}{\alpha i}\frac{\partial}{\partial\theta}e^{i\arg\left(\frac{f(z)}{z}\right)^{\alpha}}e^{-i\arg\left(\frac{g(z)}{z}\right)^{\alpha}}\mathrm{d}\theta\right|.$$

利用分部积分公式和引理 4 可得

$$J_1\leqslant 1+\frac{1}{2\pi}\int_0^{2\pi}\left\{\left|\frac{\partial}{\partial\theta}\arg g(z)\right|+\left|\frac{\partial}{\partial\theta}\arg z\right|\right\}\mathrm{d}\theta\leqslant 3.$$

(2)记 $\mathrm{Re}\frac{zf'(z)}{f(z)}=u(re^{i\theta})$,$\mathrm{Im}\frac{zf'(z)}{f(z)}=v(re^{i\theta})$,由柯西-黎曼条件可知,对于 $1/2\leqslant r<1$ 有

$$J_2\leqslant\frac{1}{2\pi}\left|\int_0^{2\pi}v\left(\frac{1}{2}e^{i\theta}\right)e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}\theta\right|+\frac{1}{2\pi}\left|\int_0^{2\pi}\int_{\frac{1}{2}}^{r}\frac{1}{r}\frac{\partial u(re^{i\theta})}{\partial\theta}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}r\mathrm{d}\theta\right|=J_{21}+J_{22}.$$

而

$$J_{21}\leqslant\frac{1}{2\pi}\int_0^{2\pi}\max_{\theta\in[0,2\pi]}\left|v\left(\frac{1}{2}e^{i\theta}\right)\right|\mathrm{d}\theta\leqslant\max_{\theta\in[0,2\pi]}\left|\frac{f'(e^{i\theta}/2)/2}{f(e^{i\theta}/2)}\right|\leqslant\frac{1+1/2}{1-1/2}=3.$$

由分部积分和引理 2 可知

$$J_{22}=\frac{\alpha}{2\pi}\left|\int_{\frac{1}{2}}^{r}\int_0^{2\pi}\frac{1}{r}u(re^{i\theta})e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\left(\frac{\partial}{\partial\theta}\arg\frac{f(z)}{z}-\frac{\partial}{\partial\theta}\arg\frac{g(z)}{z}\right)\mathrm{d}\theta\mathrm{d}r\right|.$$

由引理 1、引理 3 和 $1/2\leqslant r<1$ 可知

$$J_{22}\leqslant 4\alpha\left(1+2\log\frac{1}{1-r}\right),$$

则

$$J_2=\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{Im}\frac{zf'(z)}{f(z)}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}\mathrm{d}\theta\right|\leqslant 3+4\alpha\left(1+2\log\frac{1}{1-r}\right).$$

(3)由式(2)可知

$$\frac{zf'(z)}{f(z)}=1+z\left(\log\frac{f(z)}{z}\right)'=1+\sum_{k=1}^{\infty}2kb_kz^k,\qquad(3)$$

则

$$\frac{zf'(z)}{f(z)}e^{in\theta}=\frac{1}{i}\frac{\partial}{\partial\theta}\left(\frac{e^{in\theta}}{n}+\sum_{k=1}^{\infty}\frac{2kb_kr^ke^{i(n+k)\theta}}{n+k}\right)=\frac{1}{i}\frac{\partial}{\partial\theta}F(z).$$

由引理 1、引理 2 和分部积分可得

$$J_3\leqslant 2\alpha\left(\frac{1}{2\pi}\int_0^{2\pi}|F(z)|^2\mathrm{d}\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zf'(z)}{f(z)}\right|^2\mathrm{d}\theta\right)^{\frac{1}{2}}+2\alpha\left(\frac{1}{2\pi}\int_0^{2\pi}|F(z)|^2\mathrm{d}\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{zg'(z)}{g(z)}\right|^2\mathrm{d}\theta\right)^{\frac{1}{2}}.$$

由引理 5 可知

$$\frac{1}{2\pi}\int_0^{2\pi}|F(z)|^2\mathrm{d}\theta=\frac{1}{n^2}+4\sum_{k=1}^{\infty}\frac{k^2|b_k|^2r^{2k}}{(n+k)^2}\leqslant\frac{1}{n^2}+\frac{4}{n}\sum_{k=1}^{\infty}k^2|b_k|^2r^{2k}\leqslant\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r},$$

则

$$J_3\leqslant 4\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}.$$

引理 7　设 $f(z)\in B(\lambda,\alpha,\beta)$,$g(z)\in S^*$ 满足式(1),则对 $z=re^{i\theta}$ $(0\leqslant r<1)$,有

$$\left|\left(\frac{g(z)}{f(z)}\right)^{\alpha}\right|\leqslant\frac{1+r}{1-|1-2\beta|r}\left[1+\lambda+\lambda\frac{1+r}{1-r}\right].$$

由从属关系和 $\frac{1-r}{1+r}\leqslant\left|\frac{zf'(z)}{f(z)}\right|\leqslant\frac{1+r}{1-r}$,易证引理 7 成立.

2 主要结论

定理1 设$f(z)\in B(\lambda,\alpha,\beta)$,则对$n\geqslant 2$,有

$$|b_n|\leqslant\frac{A}{n}\log n+\frac{B}{n}+\frac{32\beta}{1-|1-2\beta|},$$

其中,$A=\left(32+8\left(\frac{1}{(2\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}\right)\alpha$,$B=24+16\alpha+8\frac{|1-\lambda|}{\lambda}+\frac{1}{\lambda(1-|1-2\beta|)}$.

证明 设$f(z)\in B(\lambda,\alpha,\beta)$,则存在$g(z)\in S^*$,使得式(1)成立. 记

$$p(z)=(1-\lambda)\left(\frac{f(z)}{g(z)}\right)^{\alpha}+\lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha}-\beta,\quad(4)$$

则 $\mathrm{Re}\,p(z)>0$. 由式(3)可知对$z=re^{i\theta}$,有

$$2nb_n=\frac{1}{2\pi i}\oint_{|z|=r}\frac{zf'(z)}{f(z)}z^{-n-1}dz=\frac{1}{2\pi}\int_0^{2\pi}\frac{zf'(z)}{f(z)}r^{-n}e^{-in\theta}d\theta.$$

由式(4)和$p(z)=2\mathrm{Re}\,p(z)-\overline{p(z)}$可知

$$\begin{aligned}|2nb_nr^n|\leqslant&\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{2}{\lambda}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\mathrm{Re}\,p(z)e^{-in\theta}d\theta\right|+\\&\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{1}{\lambda}\left(\frac{g(z)}{f(z)}\right)^{\alpha}\overline{p(z)}e^{-in\theta}d\theta\right|+\\&\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{\beta}{\lambda}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-in\theta}d\theta\right|+\frac{|1-\lambda|}{\lambda}=\\&I_1+I_2+I_3+\frac{|1-\lambda|}{\lambda}.\end{aligned}$$

由 $\mathrm{Re}\,p(z)>0$ 和式(4),可知

$$\begin{aligned}I_1\leqslant&\frac{1}{\lambda\pi}\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}\mathrm{Re}\,p(z)d\theta\leqslant\\&\frac{1}{\lambda\pi}\left|\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}p(z)d\theta\right|\leqslant\\&\frac{1}{\pi}\left|\int_0^{2\pi}\mathrm{Re}\frac{zf'(z)}{f(z)}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\\&\frac{1}{\pi}\left|\int_0^{2\pi}\mathrm{Im}\frac{zf'(z)}{f(z)}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\\&\frac{|1-\lambda|}{\lambda\pi}\left|\int_0^{2\pi}e^{i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}d\theta\right|+\frac{\beta}{\lambda\pi}\left|\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}d\theta\right|.\end{aligned}$$

由引理6可知

$$I_1\leqslant 12+8\alpha\left(1+2\log\frac{1}{1-r}\right)+2\frac{|1-\lambda|}{\lambda}+\frac{\beta}{\lambda\pi}\left|\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}d\theta\right|,$$

$$\begin{aligned}I_2\leqslant&\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{zf'(z)}{f(z)}e^{2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}e^{in\theta}d\theta\right|+\\&\frac{1}{2\pi}\left|\int_0^{2\pi}\frac{1-\lambda}{\lambda}e^{-2i\arg\left(\frac{f(z)}{g(z)}\right)^{\alpha}}e^{-in\theta}d\theta\right|+\\&\frac{\beta}{2\lambda\pi}\left|\int_0^{2\pi}\left(\frac{g(z)}{f(z)}\right)^{\alpha}e^{-in\theta}d\theta\right|\leqslant\\&4\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}+\\&\frac{|1-\lambda|}{\lambda}+\frac{\beta}{2\lambda\pi}\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}d\theta.\end{aligned}$$

由引理7可知

$$I_3\leqslant\int_0^{2\pi}\left|\frac{g(z)}{f(z)}\right|^{\alpha}d\theta\leqslant 2\pi\frac{(1+r)(1+\lambda+\lambda(1+r)/(1-r))}{1-|1-2\beta|r}.$$

所以

$$\begin{aligned}|b_n|\leqslant&\frac{1}{2n}r^{-n}\left(12+8\alpha\left(1+2\log\frac{1}{1-r}\right)+4\frac{|1-\lambda|}{\lambda}\right)+\\&\frac{1}{2n}r^{-n}\frac{4\beta(1+r)(1+\lambda+\lambda(1+r)/(1-r))}{\lambda\;\;1-|1-2\beta|r}+\\&\frac{1}{2n}r^{-n}\left(4\alpha\left(\frac{1}{n^2}+\frac{4}{n}\log\frac{1}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{4}{1-r}\log\frac{1}{1-\sqrt{r}}\right)^{\frac{1}{2}}\right).\end{aligned}$$

设$r=1-1/n$,经简单计算可得

$$|b_n|\leqslant\frac{A}{n}\log n+\frac{B}{n}+\frac{32\beta}{1-|1-2\beta|},$$

其中,$A=\left(32+8\left(\frac{1}{(2\log 2)^2}+\frac{6}{\log 2}+32\right)^{\frac{1}{2}}\right)\alpha$,$B=24+16\alpha+8\frac{|1-\lambda|}{\lambda}+\frac{1}{\lambda(1-|1-2\beta|)}$.

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